2013

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define what it means for a function $f : \mathbb{C} \to \mathbb{C}$ to be holomorphic at the point z_0.

f is holomorphic at z_0 if
$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$
exists. If it does we set
$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$
and we call this the derivative of f at z_0.

(b) Show that $f(z) = \bar{z}$ is not holomorphic at z_0 for any $z_0 \in \mathbb{C}$.

We have
$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0} = \lim_{z \to z_0} \frac{\overline{z - z_0}}{z - z_0}.$$
To show that the limit does not exist, we approch z_0 first horizontally and then vertically, i.e.

(i) $z - z_0 \in \mathbb{R}$. Then $\overline{z - z_0} = z - z_0$ so that
$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z - z_0}{z - z_0} = 1.$$

(ii) $z - z_0 \in i \cdot \mathbb{R}$, i.e. $z - z_0$ is purely imaginary. Then $\overline{z - z_0} = -(z - z_0)$ so that
$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{-(z - z_0)}{z - z_0} = -1.$$

As the two answers are different, the limit does not exist and f is not differentiable at z_0.

(c) Write the Cauchy-Riemann equations for u and v, where $f(z) = u(x, y) + iv(x, y)$ i.e. u and v are the real and imaginary part of f. Show that, if f is holomorphic, then u is harmonic. You may assume that the second partial derivatives of u and v exist and are continuous.

The Cauchy–Riemann equations are
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The function u is harmonic if $\Delta u = 0$, i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

by the Cauchy–Riemann equations. We have used that the mixed second partial derivatived are equal, which is guaranteed by the continuity of the second partial derivatives.

(d) For the harmonic function $v : \mathbb{R}^2 \setminus \{0\} \to \mathbf{R}$ given by the formula

$$v(x, y) = \frac{-y}{x^2 + y^2}$$

find all holomorphic functions $f(z)$ such that $\Im f(z) = v(x, y)$. Write f as a function of z.

We use the second of the Cauchy–Riemann equations. We have

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{-y}{x^2 + y^2}\right) = -\frac{2xy}{(x^2 + y^2)^2}.$$

We notice that this expression is symmetric with respect to x and y. This simplifies the integration below:

$$u(x, y) = \int -\frac{2xy}{(x^2 + y^2)^2}\, dy = \frac{x}{x^2 + y^2} + c(x),$$

where $c(x)$ is the constant of integration, depending possibly on x but not on y. We need to find $c(x)$. We differentiate with respect to x and use the first Cauchy–Riemann equation.

$$\frac{\partial u}{\partial x} = \frac{1(x^2 + y^2) - 2x \cdot x}{(x^2 + y^2)^2} + c'(x) = \frac{y^2 - x^2}{(x^2 + y^2)^2} + c'(x) = \frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

the last by direct calculation. We deduce that $c'(x) = 0 \Longrightarrow c(x) = k$ a constant independent of both x and y. So

$$u(x, y) = \frac{x}{x^2 + y^2} + k$$

$$f(z) = u(x, y) + iv(x, y) = \frac{x}{x^2 + y^2} + k + i\frac{-y}{x^2 + y^2} = \frac{x - iy}{x^2 + y^2} + k = \frac{\bar{z}}{|z|^2} + k = \frac{1}{z} + k.$$

Alternative solutions: We notice that $g(z) = 1/z$ is holomorphic and has $\Im g(z) = \frac{-y}{x^2+y^2}$. For any solution $f(z)$ we will have $\Im(f(z) - g(z)) = 0$, i.e. the holomorphic function $f - g$ has constant imaginary part. Therefore, by a well-known theorem, it is a constant on the region, i.e.

$$f(z) = g(z) + k \quad \forall z \in \mathbb{C} \setminus \{0\}.$$

2. (a) Assume that f is holomorphic on the domain D and that $|f(z)|$ is constant on D. Show that f is a constant function. You may use the fact that if $f'(z) = 0$ on D, then f is constant.

Let $|f(z)| = k$. Then

$$|f(z)|^2 = k^2 \Longrightarrow u^2 + v^2 = k^2. \tag{1}$$

If $k = 0$, then $u(x, y) = v(x, y) = 0 \Longrightarrow f(z) = 0$. So we can assume that $k \neq 0$. We differentiate (1) in x and y to get

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \tag{2}$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \tag{3}$$

Eq. (3) becomes, using the Cauchy–Riemann equations

$$v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} = 0. \tag{4}$$

We solve the system of (2) and (4). We multiply (2) with u and (4) with v and add to get

$$u^2\frac{\partial u}{\partial x} + v^2\frac{\partial u}{\partial x} = 0 \Longrightarrow (u^2 + v^2)\frac{\partial u}{\partial x} = 0 \Longrightarrow k^2\frac{\partial u}{\partial x} = 0.$$

Since $k \neq 0$ we get $\partial u/\partial x = 0$. By substitution we also get $\partial v/\partial x = 0$. We have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + i0 = 0.$$

Now we use the fact that $f'(z) = 0$ for all z in a domain implies that f is constant.

(b) Establish the following integration formula with the aid of residues:

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{6}.$$

Complete explanations are required.

We consider

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$$

and the contour γ in Figure 1. By Cauchy's residue theorem

$$\int_\gamma f(z)\,dz = 2\pi i \sum_j \text{res}\,(f, z_j),$$

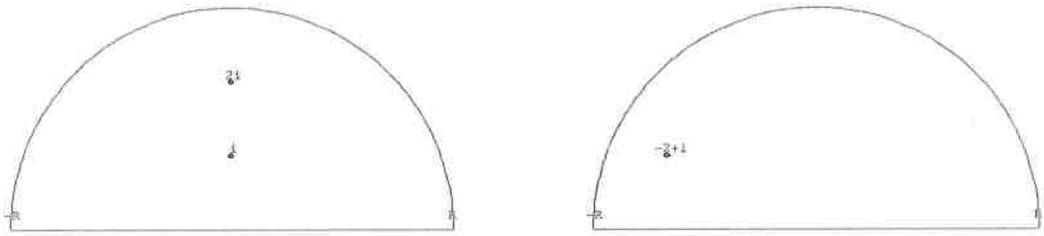

Figure 1: Contours for problem 2(b) and 5(b)

where z_j are the poles of $f(z)$ inside γ. The poles of f are at the zeros of the denominator $\pm i$ and $\pm 2i$. Only i and $2i$ are inside γ and only when $R > 2$. Since the zeros are all simple, the poles are simple. We calculate the residues.

$$\text{res }(f, i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}(z-i)\frac{z^2}{(z-i)(z+i)(z^2+4)} = \frac{i^2}{2i(i^2+4)} = \frac{-1}{2i\cdot 3} = -\frac{1}{6i}.$$

$$\text{res }(f, 2i) = \lim_{z\to 2i}(z-2i)f(z) = \lim_{z\to 2i}(z-2i)\frac{z^2}{(z-2i)(z+2i)(z^2+1)} = \frac{(2i)^2}{4i((2i)^2+1)} = \frac{-4}{4i\cdot(-3)} = \frac{1}{3i}.$$

Therefore,

$$\int_\gamma f(z)\,dz = 2\pi i\left(-\frac{1}{6i} + \frac{1}{3i}\right) = 2\pi i\frac{1}{6i} = \frac{\pi}{3}.$$

The contour γ can be split in two parts: the horizontal segment $[-R, R]$ traversed from left to right and γ_R the semicircle traversed anticlockwise. On $[-R, R]$ we have the parametrisation $z = x$, $-R \leq x \leq R$, which gives $dz = dx$ and

$$\int_{[-R,R]} f(z)\,dz = \int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = 2\int_0^R \frac{x^2}{(x^2+1)(x^2+4)}\,dx$$

as the integrand is an even function. If we show that

$$\lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz = 0$$

then

$$\lim_{R\to\infty}\left(\int_{[-R,R]} f(z)\,dz + \int_{\gamma_R} f(z)\,dz\right) = \frac{\pi}{3}$$

$$\Longrightarrow \lim_{R\to\infty} 2\int_0^R \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{3}$$
$$\Longrightarrow \int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{6}.$$

On γ_R we have $z = Re^{it}$, $|z^2+1| \geq |z|^2 - 1 = R^2 - 1$, $|z^2+4| \geq |z|^2 - 4 = R^2 - 4$ and, therefore,

$$\left|\int_{\gamma_R} f(z)\,dz\right| \leq \text{length } (\gamma_R) \max_{z\in\gamma_R} |f(z)| \leq \pi R \frac{R^2}{(R^2-1)(R^2-4)} \to 0$$

as $R \to \infty$, since the numerator is R^3 while there are four powers of R in the denominator.

3. (a) State Goursat's theorem.

Let γ be a triangular contour and f be holomorphic on an open set U containing γ and its interior. Then

$$\int_\gamma f(z)\,dz = 0.$$

(b) Let f be a holomorphic function on a domain D containing a rectangle R and its interior. Using Goursat's theorem, show that

$$\int_R f(z)\,dz = 0.$$

Let R be the rectangular contour $DCBA$ traversed anticlockwise, as in Figure 2. We draw the diagonal DB and we consider the two triangular contours DCB and BAD traversed anticlockwise. By properties of complex integration

$$\int_{BD} f(z)\,dz = -\int_{DB} f(z)\,dz.$$

Goursat's theorem gives

$$\int_{BAD} f(z)\,dz = 0 = \int_{DCB} f(z)\,dz.$$

Adding these, we see that the contribution of the diagonal cancels, as it is traversed in opposite directions. We, therefore, get

$$\int_{DCBA} f(z)\,dz = \int_R f(z)\,dz = 0.$$

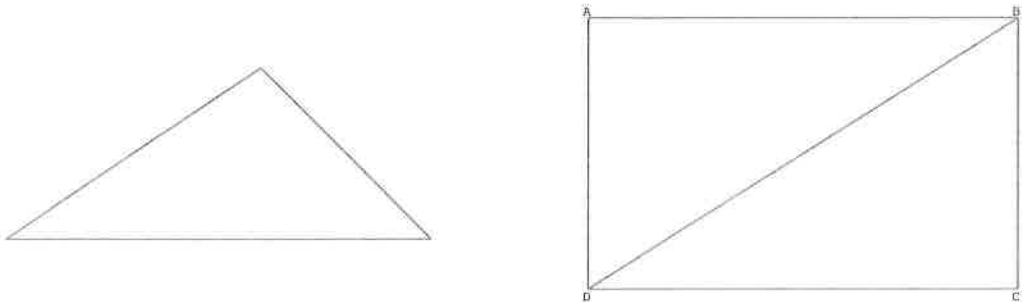

Figure 2: Contours for problem 3(a) and 3(b)

(c) Show that the map $w = \dfrac{z-i}{z+i}$ maps the upper half-plane $\{z, \Im(z) > 0\}$ conformally onto the unit disc $\{z, |z| < 1\}$.

We prove that

$$|w| < 1 \Leftrightarrow \Im(z) > 0. \tag{5}$$

$$|w| = \left|\frac{z-i}{z+i}\right| < 1 \Leftrightarrow |z-i| < |z+i| \Leftrightarrow |z-i|^2 < |z+i|^2$$

$$\Leftrightarrow (z-i)(\bar{z}+i) < (z+i)(\bar{z}-i) \Leftrightarrow z\bar{z} - i\bar{z} + iz + 1 < z\bar{z} + i\bar{z} - iz + 1$$

$$i(z-\bar{z}) < -i(z-\bar{z}) \Leftrightarrow i2i\Im(z) < -i2i\Im(z) \Leftrightarrow -2\Im(z) < 2\Im(z) \Leftrightarrow 0 < 4\Im(z) \Leftrightarrow \Im(z) > 0.$$

Geometrically the distance to i is less than the distance to $-i$ iff z is in the half-plane determined by the perpendicular bisector of the segment from i to $-i$ and containing i. The bisector is clearly the real axis.

The map is holomorphic on $\{z, \Im(z) > 0\}$ as $-i$ (root of the denominator) is not in it. It is a rational function. The inverse map is given by

$$w = \frac{z-i}{z+i} \Leftrightarrow wz + iw = z - i \Leftrightarrow z(w-1) = -i - iw = -i(w+1) \Leftrightarrow z = -i\frac{w+1}{w-1}.$$

This is also holomorphic on $D(0,1)$ as it is a rational function and $1 \notin D(0,1)$. The last calculation shows that the map is a injection. The surjectivity follows from (5).

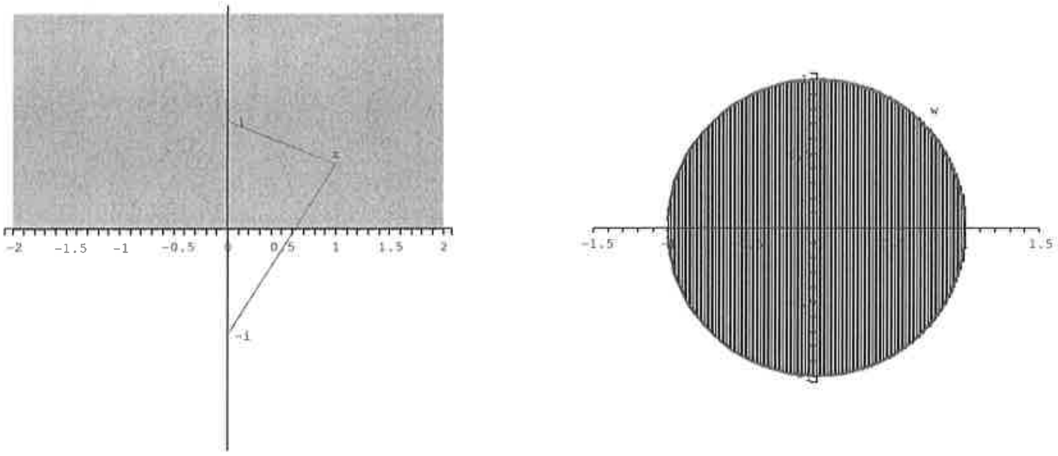

Figure 3: Regions for problem 3(c)

4. (a) State Cauchy's integral formulas for f and its derivatives.

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz,$$

where f is holomorphic on an open set U containing the closed disc $\overline{D(a,R)}$ and C is the circle centered at a with radius R traversed anticlockwise and $z_0 \in D(a,R)$.

(b) What is the value of the integral

$$\int_C \frac{1}{z^2+1}\,dz,$$

where C is (i) the circle $|z| = 2$ traversed anticlockwise, (ii) the circle $|z-i| = 1$ traversed anticlockwise?

We have

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}.$$

For (ii) we notice that $f(z) = 1/(z+i)$ i holomorphic on and inside the circle $|z-i| = 1$. So Cauchy's integral formula gives

$$\int_C \frac{1}{(z-i)(z+i)}\,dz = 2\pi i f(i) = 2\pi i \frac{1}{i+i} = \pi.$$

For (i) we use partial fractions:

$$\frac{1}{z^2+1} = \frac{1/(2i)}{z-i} + \frac{-1/(2i)}{z+i}.$$

We set $f_1(z) = 1/(2i)$. Then

$$\int_C \frac{1}{(z-i)(z+i)}\,dz = \int_C \frac{1/(2i)}{z-i}\,dz - \int_C \frac{1/(2i)}{z+i}\,dz = 2\pi i f_1(i) - 2\pi i f_1(-i) = 0,$$

as f_1 is constant. We have used Cauchy's integral formula twice.

Alternative method: Use Cauchy's residue theorem. With $f(z) = 1/(z^2+1)$ we have residues at $\pm i$. Therefore,

$$\text{res }(f, \pm i) = \lim_{z\to\pm i}(z \mp i)f(z) = \lim_{z\to\pm i}\frac{1}{z \pm i} = \frac{1}{\pm 2i}.$$

By the residue theorem

$$\int_C f(z)\,dz = 2\pi i(\text{res }(f,i) + \text{res }(f,-i)) = 0.$$

(c) Assume that f is entire and satisfies for some constant M the inequality

$$|f(z)| \le M(1+|z|)^{5/2}, \quad \forall z \in \mathbb{C}.$$

Show that f is a polynomial of degree ≤ 2.

We know that f has a Taylor expansion at 0:

$$f(z) = \sum_{n\ge 0} a_n z^n,$$

with

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

To show that f is a polynomial of degree ≤ 2, it suffices to prove that

$$a_k = 0, \quad \forall k \ge 3.$$

Cauchy's inequalities give:

$$\frac{f^{(k)}(0)}{k!} \le \frac{\max_{|z|=R}|f(z)|}{R^k} \le \frac{M(1+R)^{5/2}}{R^k} \to 0,$$

as $R \to \infty$, for $k \ge 3$ as $k > 5/2$.

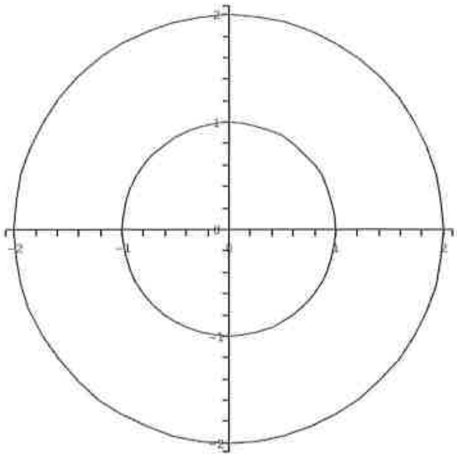

Figure 4: Region for problem 5(a)

5. (a) How many roots does the polynomial $f(z) = 2z^5 - 6z^2 + z + 1$ have inside the annulus

$$1 < |z| < 2?$$

Explain your answer.

On $|z| = 2$ we have $|2z^5| = 2 \cdot 2^5 = 64$, while

$$|-6z^2 + z + 1| \leq 6|z|^2 + |z| + 1 = 6 \cdot 2^2 + 2 + 1 = 24 + 2 + 1 = 27.$$

We notice that $64 > 27$. With $f_1(z) = 2z^5$ and $g_1(z) = -6z^2 + z + 1$, Rouché's theorem gives that $f_1 + g_1 = f$ has the same number of zeros as f_1 inside $|z| = 2$. Since $2z^5 = 0$ has root only 0 but with multiplicity 5, $f(z) = 0$ has 5 solutions inside $|z| = 2$.

Now we look inside the circle $|z| = 1$. On $|z| = 1$ we have $|-6z^2| = 6|z|^2 = 6$, while

$$|2z^5 + z + 1| \leq 2|z|^5 + |z| + 1 = 2 + 1 + 1 = 4.$$

We notice that $6 > 4$. With $f_2(z) = -6z^2$ and $g_2(z) = 2z^5 + z + 1$ Rouché's theorem gives that $f_2 + g_2 = f$ has inside $|z| = 1$ the same number of zeros as f_2. Moreover, there are no zeros of $f_2 + g_2$ on $|z| = 1$. Since $f_2(z) = 0$ has zero at 0 with multiplicity 2, $f(z) = 0$ has 2 solutions inside $|z| = 1$. We subtract the answers to get that inside the annulus $1 < |z| < 2$ $f(z) = 0$ has $5 - 2 = 3$ solutions.

(b) Establish the following integration formula with the aid of residues:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5}\, dx = -\frac{\pi \sin 2}{e}.$$

Complete explanations are required.

We consider

$$f(z) = \frac{e^{iz}}{z^2+4z+5}$$

and use the contour in Figure 1. The poles of f occur at the zeros of z^2+4z+5. We solve

$$z^2+4z+5=0 \Leftrightarrow (z+2)^2+1=0 \Leftrightarrow z+2=\pm i \Leftrightarrow z=-2\pm i.$$

Only $-2+i$ is inside the contour and only for $R>\sqrt{5}=|-2+i|$. The pole is simple as the zero is simple. We calculate the residue

$$\text{res}\,(f,-2+i) = \lim_{z\to -2+i}(z+2-i)f(z) = \lim_{z\to-2+i}\frac{e^{iz}(z+2-i)}{(z+2-1)(z+2+i)} = \frac{e^{i(-2+i)}}{-2+i+2+i}$$

$$= \frac{e^{-2i}e^{-1}}{2i} = \frac{\cos(-2)+i\sin(-2)}{2ei} = \frac{\cos 2 - i\sin 2}{2ei}.$$

We split the contour to the horizontal segment $[-R,R]$ and the semicircle γ_R. On $[-R,R]$ we have $z=x$, $-R\le x\le R$, $dz=dx$

$$\int_{[-R,R]} f(z)\,dz = \int_{-R}^{R}\frac{e^{ix}}{x^2+4x+5}\,dx = \int_{-R}^{R}\frac{\cos x}{x^2+4x+5}\,dx + i\int_{-R}^{R}\frac{\sin x}{x^2+4x+5}\,dx$$

When $R\to\infty$ we get

$$\lim_{R\to\infty}\int_{[-R,R]} f(z)\,dz = \int_{-R}^{R}\frac{e^{ix}}{x^2+4x+5}\,dx = \int_{-\infty}^{\infty}\frac{\cos x}{x^2+4x+5}\,dx + i\int_{-\infty}^{\infty}\frac{\sin x}{x^2+4x+5}\,dx,$$

so that the integral in the problem is the imaginary part of the limit. The residue theorem gives

$$\int_{[-R,R]} f(z)\,dz + \int_{\gamma_R} f(z)\,dz = 2\pi i\,\text{res}\,(f,-2+i) = 2\pi i\frac{\cos 2 - i\sin 2}{2ei} = \pi\frac{\cos 2 - i\sin 2}{e}.$$

If we show that

$$\lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz = 0,$$

then we get

$$\int_{-\infty}^{\infty}\frac{\sin x}{x^2+4x+5}\,dx = -\frac{\pi\sin 2}{e}.$$

We parametrise γ_R as $z(t)=Re^{it}$, $0\le t\le\pi$. On it, using he triangle inequality twice, we have

$$|z^2+4z+5| \ge |z^2| - |4z+5| \ge |z|^2 - (|4z|+5) = R^2-4R-5.$$

On the other hand $|e^{iz}| = |e^{iR\cos t - R\sin t}| = e^{-R\sin t} \le 1$ as $\sin t\ge 0$. So

$$\left|\int_{\gamma_R} f(z)\,dz\right| = \left|\int_{\gamma_R}\frac{e^{iz}}{z^2+4z+5}\,dz\right| \le \text{length}\,(\gamma_R)\max_{z\in\gamma_R}|f(z)| \le \pi R\frac{1}{R^2-4R-5} \to 0$$

as $R\to\infty$.

6. (a) Let $f(z)$ be holomorphic on an open set that contains the closed unit disc $\{z, |z| \le 1\}$ and $f(0) = 1/2$. By working with

$$\frac{1}{2\pi i}\int_{|z|=1}\left[2 \pm (z + \frac{1}{z})\right] f(z)\frac{dz}{z}$$

prove that

$$\frac{2}{\pi}\int_0^{2\pi} f(e^{it})\cos^2\frac{t}{2}\,dt = 1 + f'(0), \quad \frac{2}{\pi}\int_0^{2\pi} f(e^{it})\sin^2\frac{t}{2}\,dt = 1 - f'(0).$$

[ ]

We have $2\sin^2(t/2) = 1 - \cos t$ and $2\cos^2(t/2) = 1 + \cos t$. Parameterizing the circle as $z = e^{it}$ we get

$$\frac{1}{2\pi i}\int_{|z|=1}\left[2 \pm (z + \frac{1}{z})\right] f(z)\frac{dz}{z} = \frac{1}{2\pi i}\int_0^{2\pi}(2 \pm 2\cos t) f(e^{it}) i dt$$

$$= \frac{1}{\pi}\int_0^{2\pi}(1 \pm \cos t) f(e^{it}) dt = \frac{2}{\pi}\int_0^{2\pi}\left\{\begin{array}{c}\cos^2(t/2)\\ \sin^2(t/2)\end{array}\right\} f(e^{it}) dt.$$

This is how we get the two integrals on the left-hand side of the result. For the right-hand sides we use the Cauchy Integral formulas:

$$\frac{1}{2\pi i}\int_{|z|=1} f(z)\frac{dz}{z} = f(0) = 1/2, \quad \frac{1}{2\pi i}\int_{|z|=1|}\frac{f(z)}{z^2}dz = f'(0),$$

while Cauchy's theorem gives directly

$$\frac{1}{2\pi i}\int_{|z|=1} f(z)\,dz = 0.$$

As a result

$$\frac{1}{2\pi i}\int_{|z|=1}\left[2 \pm (z + \frac{1}{z})\right] f(z)\frac{dz}{z} = \frac{1}{2\pi i}\int_{|z|=1}\left(2\frac{f(z)}{z} \pm (f(z) + \frac{f(z)}{z^2})\right) dz = 1 \pm f'(0).$$

(b) Establish the following integration formula with the aid of residues:

$$\int_0^{2\pi}\frac{5}{5 + 3\cos t}\,dt = \frac{5\pi}{2}.$$

Complete explanations are required.

We use

$$f(z) = \frac{5}{5 + 3(z + z^{-1})/2}\frac{1}{iz}$$

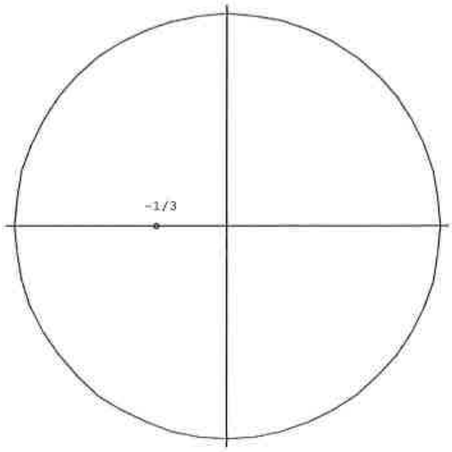

Figure 5: Contour for problem 6(b)

so that on the contour C given by $|z| = 1$ we have $z = e^{it}$, $z^{-1} = e^{-it}$, $dz = ie^{it}dt$ and

$$\int_C f(z)\,dz = \int_0^{2\pi} \frac{5}{5 + 3(e^{it} + e^{-it})/2} \frac{1}{ie^{it}} ie^{it}\,dt = \int_0^{2\pi} \frac{5}{5 + 3\cos t}\,dt.$$

We rewrite

$$f(z) = \frac{5}{5z + (3/2)(z^2 + 1)} \frac{1}{i} = \frac{10}{10z + 3z^2 + 3} \frac{1}{i}.$$

To find the root of the denominator we solve

$$3z^2 + 10z + 3 = 0 \Leftrightarrow z = \frac{-10 \pm \sqrt{100 - 4 \cdot 3 \cdot 3}}{2 \cdot 3} = \frac{-10 \pm \sqrt{64}}{6} = -3,\ -1/3.$$

Only $-1/3$ is inside C. This is a simple root, therefore, a simple pole of f. We calculate the residue

$$\text{res}\,(f, -1/3) = \lim_{z \to -1/3} (z+1/3) f(z) = \lim_{z \to -1/3} (z+1/3) \frac{10}{3(z+3)(z+1/3)} \frac{1}{i} = \frac{10}{3(-1/3+3)i}$$

$$= \frac{10}{(-1+9)i} = \frac{10}{8i} = \frac{5}{4i}.$$

The residue theorem gives

$$\int_C f(z)\,dz = 2\pi i \text{res}\,(f, -1/3) = 2\pi i \frac{5}{4i} = \frac{5\pi}{2}.$$